BIVARIATE HILBERT FUNCTIONS FOR THE TORSION FUNCTOR

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Abstract. Let (R, P) be a commutative, local Noetherian ring, I, J ideals, M and N finitely generated R-modules. Suppose $J + ann_R M + ann_R N$ is P-primary. The main result of this paper is Theorem 6, which gives necessary and sufficient conditions for the length of $\mathrm{Tor}_i(M/I^n M, N/J^m N)$, to agree with a polynomial, for m, $n \gg 0$. As a corollary, it is shown that the length of $\mathrm{Tor}_i(M/I^n M, N/I^n N))$ always agrees with a polynomial in n, for $n \gg 0$, provided $I + ann_R M + ann_R N$ is P-primary.

Introduction

Throughout this paper, unless otherwise stated, (R, P) is a commutative, Noetherian local ring with unit and I, J are (proper) ideals. Also, let M, N be finite R-modules, m, n be nonnegative integers, and let λ denote length. We would like to study the two-variable the Hilbert function $H(n, m) := \lambda(\mathrm{Tor}_i(M/I^n M, N/J^m N))$. On the one hand, we have in mind extending results on $H(n, m)$ of the authors of [BF], [KS] and [WCB], while on the other hand we seek two variable analogues of recent results concerning the Hilbert function $H(n) := \lambda(\mathrm{Tor}_i(M/I^n M, N))$. Previous work on $H(n)$ appears in [TM], [VK]) and [ET]. In fact, in [ET] it is shown that $H(n)$ agrees with a polynomial in n for n large, if we simply assume that the lengths $\lambda((\mathrm{Tor}_i(M/I^n M, N))$ are finite. Here we seek to give conditions under which $H(n, m)$ has polynomial growth for n and m sufficiently large. In some special cases, we give a degree bound on the resulting polynomials in n and m. Determining the exact degree of these polynomials seems to be a more difficult task. In the one variable case, [VK] and [ET] give upper bound estimates for the degree in general and while [ET], [DK] and [TM] determine the degree in some special cases.

In his Doctoral Thesis, Bruce Fields [BF] investigates two-variable functions of the form $\lambda(\mathrm{Tor}_i(R/I^n, R/J^m))$, where $i \geq 0$, under the assumption that $I+J$ is P-primary. For $i \geq 2$, he proves that these lengths are eventually given by polynomials in two variables. Actually,

since $\mathrm{Tor}_i(R/I^n, R/J^m) = \mathrm{Tor}_{i-1}(I^n, R/J^m) = \mathrm{Tor}_{i-2}(I^n, J^m)$ (by applying twice the shifting formula), his proof essentially shows that $\oplus_{m,n=0}^{\infty}\mathrm{Tor}_j(I^nM, J^mN)$, $j \geq 0$, is a finite, bigraded module, over a suitable polynomial ring over R, where M, N are two finite R-modules. It is then well-known that, if the lengths of homogeneous pieces of a finite bigraded module (over a suitable polynomial ring) are finite, then they are eventually given by a polynomial function (also see Notations and Conventions).

For $i = 0$ and $i = 1$, Fields only proves that polynomial growth holds under some rather restrictive conditions: he assumes that R is regular local, and that $\oplus_{m,n=0}^{\infty}(I^n \cap J^m)$ is a *finite* bigraded module over some polynomial ring in two sets of variables. This is, in general, a very strong condition on two ideals I, J. The function $\lambda(R/(I^n + J^m))$ has also been studied by Kishor Shah [KS] and William C. Brown [WCB], who give sufficient conditions for it to be given by a polynomial, for m, $n \gg 0$.

The present paper gives a characterization of those cases for which the length of $\mathrm{Tor}_i(M/I^nM, N/J^mN)$ has polynomial growth, provided the following condition is satisfied: $J + ann_RM + ann_RN$ is P-primary (see Theorem 6). It turns out that polynomial growth doesn't always hold, even in the case $i \geq 2$, as Fields' work might have suggested (see the Remark following Corollary 8). On the other hand, Proposition 3 shows that, provided $\mathrm{Tor}_i(I^nM, N/J^mN)$ has finite length, for all large m, n, its length is always given by a polynomial, without any restrictive assumption.

As a corollary to the proof of Theorem 6, under the assumption that $I+ann_RM+ann_RN$ is P-primary, we prove that $\lambda(\mathrm{Tor}_i(M/I^nM, N/I^nN))$ has *always* polynomial growth. Corollary 8 shows that, under the hypothesis that both $I + ann_RM + ann_RN$ and $J + ann_RM + ann_RN$ be P-primary, the length of $\mathrm{Tor}_i(M/I^nM, N/J^mN)$ has polynomial growth if and only if both $\mathrm{Tor}_i(M, N)$ and $\mathrm{Tor}_{i-1}(M, N)$ have finite length. Finally, when $M \otimes N$ has finite length, Theorem 9 gives the formula $\lambda(\mathrm{Tor}_i(M/I^nM, N/J^mN)) =$

$$\lambda(\mathrm{Tor}_i(M, N)) + \lambda(\mathrm{Tor}_{i-1}(I^nM, N)) + \lambda(\mathrm{Tor}_{i-1}(M, J^mN)) + \lambda(\mathrm{Tor}_{i-2}(I^nM, J^mN)),$$

which works for all $i \geq 0$, by assuming that all Tor_i with $i < 0$ are zero.

The main result of this paper shows that, at least when $J + ann_RM + ann_RN$ is P-primary, the nature of $\lambda(\mathrm{Tor}_i(M/I^nM, N/J^mN))$ is controlled by modules of the form

$I^nA \cap J^mB$. Therefore, a study of modules of this kind would deepen our understanding of $\lambda(\mathrm{Tor}_i(M/I^nM, N/J^mN))$.

Notation and Conventions

We will be using (free) resolutions of modules over several different rings. There will be resolutions of modules over R, graded resolutions of graded modules over the polynomial ring in r variables, $S_1 := R[X_1, ..., X_r]$, as well as bigraded resolutions of bigraded modules over the polynomial ring in two sets of variables, $S_2 := R[X_1, ..., X_r; Y_1, ..., Y_s]$. Unless otherwise stated, the Tor's are over R.

To further simplify notation, we denote $\mathcal{M} = \oplus_{n=0}^{\infty} M$, which is an (infinitely generated) graded module over the Rees ring $\mathcal{R}_I := \oplus_{n=0}^{\infty} I^n$. If I is generated by x_1,..., x_r, then $\mathcal{M}$ is naturally an infinitely generated S_1-graded module, via the canonical ring homomorphism $S_1 \longrightarrow \mathcal{R}_I$, given by $X_i \mapsto x_i$, for all i. The action of S_1 on $\mathcal{M}$ is given by $X_i v_k = x_i v_k$, where v_k denotes a homogeneous vector of degree k. Also, if we denote $\mathcal{IM} := \oplus_{n=0}^{\infty} I^n M$, then this is a finitely generated graded module over $\mathcal{R}_I$, and hence over S_1, as before. It follows that $\mathcal{M}/\mathcal{IM} = \oplus_{n=0}^{\infty}(M/I^nM)$ is a graded module over both $\mathcal{R}_I$ and S_1.

Similarly, if we assume $J = (y_1, ..., y_s)$, $\oplus_{m,n=0}^{\infty} I^n J^m M$ is a bigraded module over the bigraded Rees ring $\mathcal{R}_{I,J} := \oplus_{m,n=0}^{\infty} I^n J^m$, and hence over the polynomial ring S_2, via a similar map $S_2 \longrightarrow \mathcal{R}_{I,J}$.

Note that any graded free resolution over S_1 or S_2 of some graded module, is also a free resolution of that module over R.

We will be making use of the fact that, in a (bi)graded resolution of some S_1 (or S_2)-graded module, say $\mathcal{IM}$, by considering just its homogeneous part of degree k, we obtain a free resolution, over R, of the module I^kM, the k-th homogeneous component of $\mathcal{IM}$.

We will be making repeated use of the fact that, if $\mathcal{P} := \oplus_{m,n=0}^{\infty} P_{m,n}$ is a finite bigraded S_2-module, whose homogeneous pieces have finite length, then $\lambda(P_{m,n})$ is eventually given by a polynomial. In particular, $\lambda(\mathrm{Tor}_i(I^nM, J^mN))$ is eventually given by a polynomial. Indeed, we can take $\mathcal{C}$ a S_1-graded free resolution (consisting of finite free S_1-modules) of $\oplus_{n=0}^{\infty} I^n M$ and, similarly, $\mathcal{D}$ a S_1'-graded free resolution of $\oplus_{m=0}^{\infty} J^m N$, also consisting of finite free S_1'-modules. (Here, $S_1' = R[Y_1, ..., Y_s]$.) Then the modules in $\mathcal{C} \otimes_R \mathcal{D}$ have

a natural structure of $S_1 \otimes_R S_1' \cong S_2$-modules. Actually, $\mathcal{C} \otimes_R \mathcal{D}$ is a complex of finite, free, S_2-modules, whose i-th homology is $\mathrm{Tor}_i^R(\oplus_{n=0}^{\infty} I^n M, \oplus_{m=0}^{\infty} J^m N)$. Of course, this is a finitely generated bigraded S_2-module. Since the homogeneous components of this are just $\mathrm{Tor}_i^R(I^n M, J^m N)$, it follows that, if their lengths are finite, then these lengths are eventually given by a polynomial in m, n.

The main result

In an attempt to study the length of $\mathrm{Tor}_i(M/I^n M, N/J^m N)$ in as great generality as possible, we first investigate $\mathrm{Tor}_i(I^n M, N/J^m N)$. It turns out that in this case polynomial growth follows from the simplest assumption that these Tor's have finite length. The following few results are essentially given without proof, as their proofs parallel those of corresponding one-variable statements (see [ET]).

Proposition 1. *Let R be a Noetherian ring (not necessarily local), and $J \subset R$ an ideal. Let S_1 be the polynomial ring over R in r variables, and let*

$$\mathcal{C}: \qquad \mathcal{F}_2 \xrightarrow{\psi} \mathcal{F}_1 \xrightarrow{\phi} \mathcal{F}_0$$

be a graded complex of graded S_1-modules, graded by total degree. Assume that $\mathcal{F}_1$, $\mathcal{F}_0$ are finitely generated S_1-modules. Then, there is $l \geq 0$, such that, for all $m \geq l$

$$H_1(\mathcal{C} \otimes \frac{R}{J^m}) = \frac{\mathcal{U} + J^{m-l}\mathcal{V}}{\mathcal{Z} + J^{m-l}\mathcal{W}},$$

where $\mathcal{Z} \subseteq \mathcal{U}$ and $\mathcal{W} \subseteq \mathcal{V}$ are finite, graded S_1-modules.

Proof. It essentially goes as in Proposition 3 in [ET]. $\square$

Proposition 2. *Let R, S_1, J be as in Proposition 1. Let $\mathcal{T}$ be a graded S_1-module, and $\mathcal{U}$, $\mathcal{V}$, $\mathcal{W}$, $\mathcal{Z}$ be finite graded S_1-submodules of $\mathcal{T}$. Assume that $\mathcal{Z} \subseteq \mathcal{U}$, and that $\mathcal{W} \subseteq \mathcal{V}$, and denote*

$$\mathcal{L}_m := \frac{\mathcal{U} + J^m\mathcal{V}}{\mathcal{Z} + J^m\mathcal{W}}.$$

Then, if $(\mathcal{L}_m)_n$, the n-th degree homogeneous component of $\mathcal{L}_m$, has finite length for all large values of m and n, $\lambda((\mathcal{L}_m)_n)$ is eventually given by a polynomial in m and n.

Proof. It follows the same path as Lemma 2, (b) in [ET]. $\square$

Proposition 3. *Let R be a Noetherian ring, I, $J \subseteq R$ ideals, M, N be finite R-modules, and $i \geq 0$. If $\mathrm{Tor}_i(I^n M, N/J^m N)$ has finite length for all m, $n \gg 0$, then this length is eventually given by a polynomial in m, n.*

Proof. Take an S_1-graded resolution by finite free S_1-modules of the finite graded S_1-module $\oplus_{n=0}^{\infty} I^n M$. Tensor it with $N/J^m N$, in two steps, first with N, (call the resulting S_1-complex $\mathcal{C}$), then with R/J^m. The part giving $\mathrm{Tor}_i^R(\oplus_{m=0}^{\infty} I^n M, N/J^m N)$, looks just like the situation described in Proposition 1. Therefore, by Proposition 1, we see that

$$\mathrm{Tor}_i^R(\oplus_{n=0}^{\infty} I^n M, N/J^m N) = \frac{\mathcal{U} + J^{m-l}\mathcal{V}}{\mathcal{Z} + J^{m-l}\mathcal{W}},$$

for some l, all $m \geq l$, where $\mathcal{U}$, $\mathcal{V}$, $\mathcal{Z}$ and $\mathcal{W}$ are all finite graded S_1-modules. It follows that

$$\mathrm{Tor}_i^R(I^n M, N/J^m N) = \frac{\mathcal{U}_n + J^{m-l}\mathcal{V}_n}{\mathcal{Z}_n + J^{m-l}\mathcal{W}_n},$$

by looking at homogeneous pieces of degree n in the previous Tor formula. Thus, the conclusion follows from Proposition 2. $\square$

Lemma 4. *Let (R, P) be Noetherian, local, I, $J \subset R$ ideals, $i \geq 0$. Then, for two finite R-modules M, N, we have:*

(a) *The image of the induced map*

$$\mathrm{Tor}_i(I^n M, N) \xrightarrow{H(f_i)} \mathrm{Tor}_i(M, N)$$

is of the form $I^{n-k}A$, for some $k \geq 0$ and $n \geq k$, where A is the image of the map $\mathrm{Tor}_i(I^k M, N) \xrightarrow{H(f_i)} \mathrm{Tor}_i(M, N)$.

(b) *The image of the induced map*

$$\mathrm{Tor}_i(M, N) \xrightarrow{H(g_i)} \mathrm{Tor}_i(M, N/J^m N)$$

has the form

$$\frac{\operatorname{Tor}_i(M,N)+J^mB}{J^mB},$$

for some module B, such that $\operatorname{Tor}_i(M,N) \subseteq B$.

Proof. (a) Let

$$\cdots \longrightarrow R^{\beta_{i+1}} \longrightarrow R^{\beta_i} \longrightarrow R^{\beta_{i-1}} \longrightarrow \cdots \tag{1}$$

be a free resolution of N. Then we have the following commutative diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & I^nM^{\beta_{i+1}} & \xrightarrow{\psi_n} & I^nM^{\beta_i} & \xrightarrow{\phi_n} & I^nM^{\beta_{i-1}} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle f_i} & & \downarrow & & \\
\cdots & \longrightarrow & M^{\beta_{i+1}} & \xrightarrow{\psi} & M^{\beta_i} & \xrightarrow{\phi} & M^{\beta_{i-1}} & \longrightarrow & \cdots .
\end{array}$$

Let $K=\ker\phi$ and $L=\operatorname{im}\psi$, so $\operatorname{Tor}_i(M,N)=K/L$. We also have that $\ker\phi_n = K\cap I^nM^{\beta_i}$ and $\operatorname{im}\psi_n = I^nL$, and thus $\operatorname{Tor}_i(I^nM,N)=(K\cap I^nM^{\beta_i})/I^nL$. It follows that

$$\operatorname{im}(H(f_i)) = \frac{K\cap I^nM^{\beta_i}+L}{L} = \frac{I^{n-k}(K\cap I^kM^{\beta_i})+L}{L},$$

for some k and all $n\geq k$. Note that this is of the form $I^{n-k}A$, where A is the image of the map $\operatorname{Tor}_i(I^kM,N) \xrightarrow{H(f_i)} \operatorname{Tor}_i(M,N)$, as stated.

(b) Now assume that (1) gives a free resolution of M, and tensor it with N/J^mN. We get

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & N^{\beta_{i+1}} & \xrightarrow{\psi} & N^{\beta_i} & \xrightarrow{\phi} & N^{\beta_{i-1}} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle g_i} & & \downarrow & & \\
\cdots & \longrightarrow & N^{\beta_{i+1}}/J^mN^{\beta_{i+1}} & \xrightarrow{\psi_m} & N^{\beta_i}/J^mN^{\beta_i} & \xrightarrow{\phi_m} & N^{\beta_{i-1}}/J^mN^{\beta_{i-1}} & \longrightarrow & \cdots
\end{array}$$

Again, if we denote $K=\ker\phi$ and $L=\operatorname{im}\psi$, then $\operatorname{Tor}_i(M,N)=K/L$ and, moreover, we obtain that

$$\ker\phi_m = \frac{K+J^{m-l}(\phi^{-1}(J^lN^{\beta_{i-1}}))}{J^mN^{\beta_i}},$$

for some l and $m \geq l$.

We also get

$$\operatorname{im}\psi_m = \frac{L + J^m N^{\beta_i}}{J^m N^{\beta_i}},$$

so

$$\operatorname{Tor}_i(M, N/J^m N) = \frac{K + J^{m-l}(\phi^{-1}(J^l N^{\beta_{i-1}}))}{L + J^m N^{\beta_i}}.$$

It follows that

$$\operatorname{im} H(g_i) = \frac{K + J^m N^{\beta_i}}{L + J^m N^{\beta_i}} \cong \frac{\operatorname{Tor}_i(M,N) + J^m B}{J^m B},$$

where $B = N^{\beta_i}/L$. Of course, $\operatorname{Tor}_i(M,N) \subseteq B$. $\square$

The next Proposition is an extended version of the following well-known result: Let (R, P) be Noetherian, local, and $I \subseteq R$ an ideal. If L, M are finitely generated modules, L of finite length, then, for any $i \geq 0$, the natural map $\operatorname{Tor}_i(I^n M, L) \longrightarrow \operatorname{Tor}_i(M, L)$ is zero, for $n \gg 0$ (see [GL]).

Proposition 5. *Let (R, P) be a Noetherian, local ring. Let $I \subset R$ be an ideal, M, N two finite R-modules and $i \geq 0$, fixed. Then the following are equivalent:*

(a) $I \subseteq rad(ann_R \operatorname{Tor}_i(M,N))$.
(b) $I \subseteq rad(ann_R \operatorname{Tor}_i(I^k M,N))$*, for some* $k \geq 0$.
(c) $I \subseteq rad(ann_R \operatorname{Tor}_i(I^n M,N))$*, for all* $n \geq 0$.
(d) $I \subseteq rad(ann_R \operatorname{im}(\operatorname{Tor}_i(I^n M,N) \longrightarrow \operatorname{Tor}_i(M,N)))$*, for all* $n \geq 0$.
(e) $\operatorname{im}(\operatorname{Tor}_i(I^n M,N) \longrightarrow \operatorname{Tor}_i(M,N)) = 0$*, for all* $n \gg 0$.

Proof. Clearly, (c) implies (a) and (b). Conversely, consider the long exact sequence

$$\cdots \longrightarrow \operatorname{Tor}_{i+1}(M/I^n M, N) \xrightarrow{\partial} \operatorname{Tor}_i(I^n M, N) \xrightarrow{\alpha} \operatorname{Tor}_i(M,N) \xrightarrow{\beta} \operatorname{Tor}_i(M/I^n M, N) \longrightarrow \cdots$$

(a) implies (b), (c) follows by considering α and ∂, since $I \subseteq rad(ann_R \operatorname{Tor}_j(M/I^n M, N))$ for all $n \geq 0$. (b) implies (a) follows from (c) implies (a).
(a) implies (d) and (d) implies (a) are immediate, considering α.
(e) implies (a): if $\alpha = 0$, then β is an injection, so the conclusion follows.
(a) implies (e) follows from Lemma 4(a). $\square$

Here is the main result of this paper:

Theorem 6. *Let (R, P) be Noetherian, local, I, $J \subseteq R$ two ideals, M, N finitely generated R-modules, $i \geq 0$. Assume that $ann_R M + ann_R N + J$ is P-primary. Then,*

$$\lambda(\mathrm{Tor}_i(M/I^n M, N/J^m N))$$

is eventually given by a polynomial in m and n if and only if $I \subseteq rad(ann_R \mathrm{Tor}_j(M, N))$, for $j \in \{i-1, i\}$.

Proof. Consider the long exact sequence

$$\cdots \to \mathrm{Tor}_i(I^n M, N/J^m N) \xrightarrow{\alpha_i^{m,n}} \mathrm{Tor}_i(M, N/J^m N) \to \mathrm{Tor}_i(M/I^n M, N/J^m N) \to$$

$$\mathrm{Tor}_{i-1}(I^n M, N/J^m N) \xrightarrow{\alpha_{i-1}^{m,n}} \mathrm{Tor}_{i-1}(M, N/J^m N) \to \cdots$$

We already know that the lengths of the modules above, save the one in the middle, are (eventually) given by polynomials in one or two variables (see Proposition 3). Thus, we have

$$\lambda(\mathrm{Tor}_i(M/I^n M, N/J^m N)) = [\lambda(\mathrm{Tor}_i(M, N/J^m N)) - \lambda(\mathrm{im}\,\alpha_i^{m,n})] + \lambda(\ker \alpha_{i-1}^{m,n}) \tag{2}$$
$$= [\lambda(\mathrm{Tor}_i(M, N/J^m N)) - \lambda(\mathrm{im}\,\alpha_i^{m,n})] + [\lambda(\mathrm{Tor}_{i-1}(I^n M, N/J^m N)) - \lambda(\mathrm{im}\,\alpha_{i-1}^{m,n})].$$

Therefore, we need to examine $\lambda(\mathrm{im}\,\alpha_j^{m,n})$, for $j \in \{i-1, i\}$. Consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Tor}_i(I^n M, N) \arrow[r, "\psi^{m,n}"] \arrow[d, "\sigma^{m,n}"] & \mathrm{Tor}_i(I^n M, N/J^m N) \arrow[r, "\phi^{m,n}"] \arrow[d, "\alpha_i^{m,n}"] & \mathrm{Tor}_{i-1}(I^n M, J^m N) \arrow[d, "\tau^{m,n}"] \\
\mathrm{Tor}_i(M, N) \arrow[r, "\theta^{m,n}"] & \mathrm{Tor}_i(M, N/J^m N) \arrow[r] \arrow[d, "\pi_i^{m,n}"] & \mathrm{Tor}_{i-1}(M, J^m N) \\
& \mathrm{Tor}_i(M, N/J^m N)/\alpha_i^{m,n}(L_{m,n}) \arrow[d] & \\
& 0 &
\end{tikzcd} \tag{3}$$

where, $L_{m,n} = \mathrm{im}\,\psi^{m,n} = \ker \phi^{m,n}$.

Note that the commutative diagram (3) is a homogeneous piece of the diagram (3′) below. That's because Tor_i^R is additive, and the natural maps in (3) commute with the action of I and J on the modules occurring in this diagram. It follows that the diagram (3′) is a commutative diagram of bigraded S_2-modules and maps.

$$\begin{tikzcd}
\mathrm{Tor}_i^R(\mathcal{IM},\mathcal{N}) \arrow[r, "\psi"] \arrow[d, "\sigma"] & \mathrm{Tor}_i^R(\mathcal{IM},\mathcal{N}/\mathcal{JN}) \arrow[r, "\phi"] \arrow[d, "\alpha_i"] & \mathrm{Tor}_{i-1}^R(\mathcal{IM},\mathcal{JN}) \arrow[d, "\tau"] \\
\mathrm{Tor}_i^R(\mathcal{M},\mathcal{N}) \arrow[r, "\theta"] & \mathrm{Tor}_i^R(\mathcal{M},\mathcal{N}/\mathcal{JN}) \arrow[r] \arrow[d, "\pi_i"] & \mathrm{Tor}_{i-1}^R(\mathcal{M},\mathcal{JN}) \\
 & \mathrm{Tor}_i^R(\mathcal{M},\mathcal{N}/\mathcal{JN})/\alpha_i(\mathcal{L}) \arrow[d] & \\
 & 0 & ,
\end{tikzcd} \tag{3'}$$

where $\mathcal{L} = \oplus_{m,n=0}^{\infty} L_{m,n}$.

Observe now that $\pi_i \circ \alpha_i$ factors through the image of ϕ, which is a finitely generated, bigraded S_2-module (since $\mathrm{Tor}_{i-1}^R(\mathcal{IM},\mathcal{JN})$ is so), hence $\mathrm{im}\,(\pi_i \circ \alpha_i)$ is a finite, bigraded S_2-module. Then $\lambda(\mathrm{im}\,(\pi_i \circ \alpha_i)^{m,n})$ is eventually given by a polynomial, by classical theory.

Note that $\lambda(\mathrm{im}\,\alpha_i^{m,n}) = \lambda(\mathrm{im}\,(\pi_i \circ \alpha_i)^{m,n}) + \lambda(\alpha_i^{m,n}(L_{m,n}))$, and a similar equality holds for $i-1$ in place of i. From (2) and what we have just seen, it follows that $\lambda(\mathrm{Tor}_i(M/I^nM, N/J^mN))$ is eventually given by a polynomial, if and only if the same is true of $\lambda(\alpha_{i-1}^{m,n}(L_{m,n})) + \lambda(\alpha_i^{m,n}(L_{m,n}))$.

We now examine $\lambda(\alpha_i^{m,n}(L_{m,n}))$. From (3), we find that

$$\alpha_i^{m,n}(L_{m,n}) = \alpha_i^{m,n}(\psi^{m,n}(\mathrm{Tor}_i(I^nM,N))) = (\theta\circ\sigma)^{m,n}(\mathrm{Tor}_i(I^nM,N)). \tag{4}$$

From Lemma 4, (a) and (b), we get that

$$(\theta\circ\sigma)^{m,n}(\mathrm{Tor}_i(I^nM,N)) = \frac{I^{n-k}A + J^mB}{J^mB} = \frac{I^{n-k}A}{I^{n-k}A \cap J^mB}, \tag{5}$$

for some $k \geq 0$ and $n \geq k$, where $A = \mathrm{im}\,(\mathrm{Tor}_i(I^kM,N) \longrightarrow \mathrm{Tor}_i(M,N))$.

We now claim that $\lambda(I^{n-k}A/I^{n-k}A \cap J^mB)$ is identically zero, for m, $n \gg 0$ if and only if it is polynomial for $m,n \gg 0$, if and only if $I \subseteq rad(ann_R\mathrm{Tor}_i(M,N))$. To prove

this claim, assume $I \subseteq rad(ann_R \mathrm{Tor}_i(M,N))$. Then $I^{n-k}A = 0$, for large n, and so $\lambda(I^{n-k}A/I^{n-k}A \cap J^m B) = 0$, hence polynomial, for $n \gg 0$ and all m. It remains to check that, if $I \nsubseteq rad(ann_R \mathrm{Tor}_i(M,N))$, then $\lambda(I^{n-k}A/I^{n-k}A \cap J^m B)$ is nonzero and not given by a polynomial, for all $m, n \gg 0$. Indeed, by Proposition 5, (1) $\Leftrightarrow$ (3), we know that $I \nsubseteq rad(ann_R \mathrm{im}\,(\mathrm{Tor}_i(I^n M, N) \longrightarrow \mathrm{Tor}_i(M,N)))$, for all n, so $I^{n-k}A \neq 0$ for all $n \geq k$.

Now, since $ann_R M + ann_R N + J$ is P-primary, there is a $l \geq 0$, such that $I^l \subseteq ann_R M + ann_R N + J$. It follows that, for $n \geq lm + k$, we have

$$I^{n-k} \subseteq J^m + ann_R M + ann_R N,$$

so

$$I^{n-k}A \subseteq J^m A \subseteq J^m B,$$

since we know that $A \subseteq B$.

Thus, for $n \geq lm + k$, l and k fixed, $\lambda(I^{n-k}A/I^{n-k}A \cap J^m B)$ vanishes. On the other hand, note that, for every $n \geq k$, $I^{n-k}A/I^{n-k}A \cap J^m B \neq 0$, for all $m \gg 0$. This is so since, for every $n \geq k$, n fixed, $I^{n-k}A \cap J^m B \subsetneq I^{n-k}A$ for all large m, by Krull's Intersection Theorem. Hence $\lambda(I^{n-k}A/I^{n-k}A \cap J^m B) \neq 0$, for every $n \geq k$ and $m \gg 0$. This proves the claim, since we proved that, above the line $d : n = lm + k$ in the (m,n)-plane, $\lambda(I^{n-k}A/I^{n-k}A \cap J^m B)$ always vanishes, for large m and n, while below this line, the length in question is nonzero, in case $I \nsubseteq rad(ann_R \mathrm{Tor}_i(M,N))$.

Finally, note that both terms of the form $\lambda(I^{n-k}A/I^{n-k}A \cap J^m B)$ occurring in the formula (2) of $\lambda(\mathrm{Tor}_i(M/I^n M, N/J^m N))$ (also see (4) and (5)), actually occur with the same sign. By the claim, it follows that the sum of these two terms vanishes for all large m and n, if $I \subseteq rad(ann_R \mathrm{Tor}_i(M,N)) \cap rad(ann_R \mathrm{Tor}_{i-1}(M,N))$. On the other hand, if $I \nsubseteq rad(ann_R \mathrm{Tor}_i(M,N)) \cap rad(ann_R \mathrm{Tor}_{i-1}(M,N))$, then the sum in question vanishes above both lines $d : n = lm + k$, $d' : n = l'm + k'$, (one line for each term), but it is nonzero below both these lines, d and d'. This means that $\lambda(\mathrm{Tor}_i(M/I^n M, N/J^m N))$ can only then be (eventually) polynomial, when both terms of the form $\lambda(I^{n-k}A/I^{n-k}A \cap J^m B)$ vanish. And this happens if and only if $I \subseteq rad(ann_R \mathrm{Tor}_j(M,N))$, for $j \in \{i-1, i\}$, as stated. $\square$

The proof of Theorem 6 yields the following interesting corollary:

Corollary 7. *Let (R, P) be Noetherian, local, I an ideal M, N two finite R-modules and $i \geq 0$. Assume that $I + ann_R M + ann_R N$ is P-primary. Then*

$$\lambda(\mathrm{Tor}_i(M/I^n M, N/I^n N))$$

is given by a polynomial, for $n \gg 0$.

Proof. Note that, by the proof of Theorem 6, we only have to look at each of the two (similar) terms in $\lambda(\mathrm{Tor}_i(M/I^n M, N/J^m N))$, that turned out not to be polynomial, in general. If in each of them we set $J = I$ and $m = n$, we get two terms, each of which looks like

$$\lambda(\frac{I^{n-k}A}{I^{n-k}A \cap I^n B}).$$

It is immediate, by the Artin-Rees Lemma, that $\oplus_{n=0}^{\infty} I^{n-k}A/I^{n-k}A \cap I^n B$ is a finite graded module over the Rees ring $\mathcal{R}_I = \oplus_{n=0}^{\infty} I^n$, hence the conclusion. □

Corollary 8. *Assume that both $I + ann_R M + ann_R N$ and $J + ann_R M + ann_R N$ are P-primary, in the statement of Theorem 6. Then $\lambda(\mathrm{Tor}_i(M/I^n M, N/J^m N))$ is eventually given by a polynomial if and only if $\mathrm{Tor}_j(M, N)$ has finite length for both $j = i$, $j = i - 1$.*

Proof. $\lambda(\mathrm{Tor}_i(M/I^n M, N/J^m N))$ is eventually given by a polynomial if and only if $I \subseteq rad(ann_R \mathrm{Tor}_j(M, N))$, for $j \in \{i - 1, i\}$, if and only if $I + ann_R M + ann_R N \subseteq rad(ann_R \mathrm{Tor}_j(M, N))$, for $j \in \{i - 1, i\}$, if and only if $\mathrm{Tor}_j(M, N)$, has finite length for both $j = i - 1$ and $j = i$. □

Remark. From this corollary alone we could construct numerous examples in which $\lambda(\mathrm{Tor}_i(M/I^n M, N/J^m N))$ is not eventually polynomial. It suffices to take I and J to be P-primary ideals and M, N two finite R-modules with at least one of the two modules $\mathrm{Tor}_i(M, N)$ and $\mathrm{Tor}_{i-1}(M, N)$ not having finite length. Let us give two such examples of $\mathrm{Tor}_i(M/I^n M, N/J^m N)$ that have non-polynomial length, the second of which works for any value of i.

First, assume that R has positive depth and dimension at least two. Take $x_1, x_2, \ldots, x_t$, $t \geq 1$ to be a regular sequence, such that the ideal generated by these elements is *not* P-primary. Take $M = R/(x_1, \ldots, x_t)^s$ and $N = R/(x_1, \ldots, x_t)^r$ for some $s \geq r \geq 1$. Then

$\mathrm{Tor}_1(M,N) = (x_1,\dots,x_t)^s/(x_1,\dots,x_t)^{s+r}$ has finite length if and only if $R/(x_1,\dots,x_t)$ has finite length. This is so because, by Rees' theorem, $(x_1,\dots,x_t)^j/(x_1,\dots,x_t)^{j+1}$ is a free $R/(x_1,\dots,x_t)$-module, for all $j \geq 0$. Therefore $\mathrm{Tor}_1(M,N)$ can't have finite length by the choice of the regular sequence. Now take I and J any two P-primary ideals: by Corollary 8, the length of $\mathrm{Tor}_i(M/I^nM, N/J^mN)$ is *not* given by a polynomial, for $i \in \{1,2\}$.

Secondly, assume that R is neither regular, nor an isolated singularity. Then R_Q is *not* regular for some non-maximal prime Q. Take M and N to be any two finite R-modules, such that their annihilator is Q. Note that both M_Q and N_Q are direct sums of copies of the residue field of R_Q. Then $\mathrm{Tor}_i(M,N)$ cannot have finite length for any i. (For $i \geq 1$: this would imply that the localization at Q of $\mathrm{Tor}_i(M,N)$ vanishes, giving that R_Q is regular, contrary to the choice of R.) Now, Corollary 8 says that for any choice of two primary ideals I and J, the length of $\mathrm{Tor}_i(M/I^nM, N/J^mN)$ is not polynomial for *all* $i \geq 0$.

Theorem 9. *Let (R,P) be Noetherian local, I, $J \subseteq R$ ideals, M, N finite R-modules and $i \geq 0$. Assume that $M \otimes N$ has finite length. Then*

$$\lambda(\mathrm{Tor}_i(M/I^nM, N/J^mN))$$

is given by a polynomial, for m, $n \gg 0$.
Moreover, $\lambda(\mathrm{Tor}_i(M/I^nM, N/J^mN)) =$

$$\lambda(\mathrm{Tor}_i(M,N)) + \lambda(\mathrm{Tor}_{i-1}(I^nM,N)) + \lambda(\mathrm{Tor}_{i-1}(M,J^mN)) + \lambda(\mathrm{Tor}_{i-2}(I^nM,J^mN))$$

Proof. The first statement follows immediately from Theorem 6, since, trivially, its hypotheses are met. For the last statement, let's observe that, there is a $k \geq 0$, such that, for all $m \geq 0$, and $n \geq k$, $\sigma^{m,n}$ in (3) is the zero map, by Proposition 5. It follows that $\alpha_i^{m,n}(\operatorname{im}\psi^{m,n}) = \alpha_i^{m,n}(\ker\phi^{m,n}) = 0$, hence $\alpha_i^{m,n}$ factors through $\operatorname{im}\phi^{m,n}$, and thus (as before) $\lambda(\operatorname{im}\alpha_i^{m,n})$ is eventually given by a polynomial in m, n. Finally, by Proposition 5 again, we see that for each *fixed* m, $\operatorname{im}(\alpha_i^{m,n})$ vanishes for $n \gg 0$. Therefore, $\operatorname{im}(\alpha_i^{m,n})$ is identically zero, for all large m and n.

We also have the long exact sequence

$$\cdots \to \mathrm{Tor}_i(I^nM, N/J^mN) \xrightarrow{\alpha_i^{m,n}} \mathrm{Tor}_i(M, N/J^mN) \to \mathrm{Tor}_i(M/I^nM, N/J^mN) \to$$

$$\operatorname{Tor}_{i-1}(I^nM, N/J^mN) \xrightarrow{\alpha_{i-1}^{m,n}} \operatorname{Tor}_{i-1}(M, N/J^mN) \to \cdots ,$$

and we now know that $\alpha_i^{m,n} = \alpha_{i-1}^{m,n} = 0$ for m, $n \gg 0$. Then,

$$\lambda(\operatorname{Tor}_i(M/I^nM, N/J^mN)) = \lambda(\operatorname{Tor}_i(M, N/J^mN)) + \lambda(\operatorname{Tor}_{i-1}(I^nM, N/J^mN)). \tag{6}$$

We apply this trick two more times. We have

$$\ldots \to \operatorname{Tor}_i(M, J^mN) \xrightarrow{0} \operatorname{Tor}_i(M, N) \to \operatorname{Tor}_i(M, N/J^mN) \to$$

$$\operatorname{Tor}_{i-1}(M, J^mN) \xrightarrow{0} \operatorname{Tor}_{i-1}(M, N) \to \cdots , \tag{7}$$

where the maps marked as 0 are so by Proposition 5. We get that

$$\lambda(\operatorname{Tor}_i(M, N/J^mN)) = \lambda(\operatorname{Tor}_i(M, N)) + \lambda(\operatorname{Tor}_{i-1}(M, J^mN)). \tag{8}$$

Replacing M by I^nM in (7) and using the fact that $\oplus_{m,n=0}^{\infty}\operatorname{Tor}_i(I^nM, J^mN)$ is a finite bigraded S_2-module, we see that the maps marked as 0 will remain so, for every n and large m, again by Proposition 5. We then get that

$$\lambda(\operatorname{Tor}_{i-1}(I^nM, N/J^mN)) = \lambda(\operatorname{Tor}_{i-1}(I^nM, N)) + \lambda(\operatorname{Tor}_{i-2}(I^nM, J^mN)). \tag{9}$$

Putting together (6), (8) and (9), we obtain

$$\lambda(\operatorname{Tor}_i(M/I^nM, N/J^mN))$$

$$= \lambda(\operatorname{Tor}_i(M, N)) + \lambda(\operatorname{Tor}_{i-1}(I^nM, N)) + \lambda(\operatorname{Tor}_{i-1}(M, J^mN)) + \lambda(\operatorname{Tor}_{i-2}(I^nM, J^mN)),$$

as stated. $\square$

Note that this also yields a direct proof of the first statement of this theorem, since the four terms on the right-hand side of the equality above are eventually given by polynomials, by classical theory of finite (bi)graded modules.

Finally, we give an upper bound for the degree of the polynomial that arises in Corollary 8. Note that this estimate also applies to the case of Theorem 9.

Proposition 10. *Assume the hypotheses in Corollary 8 and suppose that the length of* $\mathrm{Tor}_i(M/I^nM, J^mN)$ *is given by a polynomial, for* m, $n \gg 0$. *Then*

$$\deg \lambda(\mathrm{Tor}_i(M/I^nM, J^mN)) \leq \ell_M(I) + \ell_N(J) - 2.$$

Proof. This is a rather crude estimate, based on the one-variable case. We simply apply Corollary 4 in [ET], separately, for fixed, large enough values of m and n, then add. For the exact degree in some special cases (in one variable, though), see [TM]. $\square$

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